- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

  Show that if x, y, and z is each a positive real number, then
  $$\frac{x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3}{x^2 \cdot y^2 \cdot z^2} \geq \frac{x^3 + y^3 + z^3 + 3x \cdot y \cdot z}{2}.$$

- **5448:** *Proposed by Yubal Barrios and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

  Evaluate: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\sum_{\substack{0\le i,j\le n\\ i+j=n}} \binom{2i}{i}\binom{2j}{j}}.$

- **5449:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the use of a computer, find the real roots of the equation
  $$x^6 - 26x^3 + 55x^2 - 39x + 10 = (3x-2)\sqrt{3x-2}.$$

- **5450:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let k be a positive integer. Calculate
  $$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dxdy,$$
  where $\lfloor a \rfloor$ denotes the floor (the integer part) of a.

Solutions

5427: *Proposed by Kenneth Korbin, New York, NY*

Rationalize and simplify the fraction

$$\frac{(x+1)^4}{x(2016x^2 - 2x + 2016)} \quad \text{if} \quad x = \frac{2017 + \sqrt{2017 - \sqrt{2017}}}{2017 - \sqrt{2017 - \sqrt{2017}}}.$$

Solution 1 by David E. Manes, SUNY at Oneonta, Oneonta, NY

Let $F = (x+1)^4/(x(2016x^2 - 2x + 2016))$ and let $y = \sqrt{2017 - \sqrt{2017}}$. Then $y^2 = 2017 - \sqrt{2017}$ and $y^4 = 2017(2018 - 2\sqrt{2017})$. Moreover,

$$x = \frac{2017+y}{2017-y}, \quad \frac{1}{x} = \frac{2017-y}{2017+y}, \quad x+1 = \frac{2(2017)}{2017-y} \quad \text{and}$$

$$x^2+1=\Big(\frac{2017+y}{2017-y}\Big)^2+1=\frac{(2017+y)^2+(2017-y)^2}{(2017-y)^2}$$
$$=\frac{2(2017^2+y^2)}{(2017-y)^2}.$$

Therefore,

$$\begin{aligned}2016(x^2+1)-2x&=2\Big[\frac{2016(2017^2+y^2)}{(2017-y)^2}-\frac{2017+y}{2017-y}\Big]\\&=2\Big[\frac{2016(2017^2+y^2)-(2017^2-y^2)}{(2017-y)^2}\Big]\\&=2\Big[\frac{2015\cdot 2017^2+2017y^2}{(2017-y)^2}\Big]\\&=2(2017)\Big[\frac{2015(2017)+y^2}{(2017-y)^2}\Big]\end{aligned}$$

Substituting these values into the fraction F and simplifying, we obtain

$$F=\frac{\left(\frac{2(2017)}{2017-y}\right)^4(2017-y)}{(2017+y)(2(2017)\left(\frac{2015(2017)+y^2}{(2017-y)^2}\right)}$$

$$=\frac{(2(2017))^3}{(2017^2-y^2)(2015\cdot 2017+y^2)}$$
$$=\frac{8(2017)^3}{2015\cdot 2017^3+2\cdot 2017(2017-\sqrt{2017})-2017(2018-2\sqrt{2017})}$$

$$=\frac{8(2017)^2}{2015\cdot 2017^2+2016}$$

$$=\frac{32546312}{8197604351}$$

$$\approx 0.003\,970\,222\,349.$$

Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

For notational convenience we set $d=2017-\sqrt{2017}$, $y=2017+\sqrt{d}$, and $z=2017-\sqrt{d}$. Thus our x is y/z. We have

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)}=\frac{\left(\frac{y}{z}+1\right)^4}{\left(\frac{y}{z}\right)\left(2016\left(\frac{y}{z}\right)^2-2\left(\frac{y}{z}\right)+2016\right)}\cdot\frac{z^4}{z^4}$$
$$=\frac{(y+z)^4}{yz(2016y^2-2yz+2016z^2)}$$

Now

$$y+z=2\cdot 2017,$$

$$\begin{aligned} yz &= 2017^2 - d \\ &= 2017^2 - 2017 + \sqrt{2017} \\ &= 2017 \cdot 2016 + \sqrt{2017}, \end{aligned}$$

and

$$\begin{aligned} 2016y^2 - 2yz + 2016z^2 &= 2016(y^2 + z^2) - 2yz \\ &= 2016((y+z)^2 - 2yz) - 2yz \\ &= 2016(y+z)^2 - 2 \cdot 2017yz \\ &= 2016(2 \cdot 2017)^2 - 2 \cdot 2017(2017 \cdot 2016 + \sqrt{2017}) \\ &= 2 \cdot 2017(2017 \cdot 2016 - \sqrt{2017}). \end{aligned}$$

Hence

$$\begin{aligned} \frac{(y+z)^4}{yz(2016y^2 - 2yz + 2016z^2)} &= \frac{2^4 \cdot 2017^4}{2 \cdot 2017(2017^2 \cdot 2016^2 - 2017)} \\ &= \frac{2^3 \cdot 2017^2}{2017 \cdot 2016^2 - 1} \\ &= \frac{32546312}{8197604351}. \end{aligned}$$

Solution 3 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Let $y = 2017$ and $w = \sqrt{y - \sqrt{y}}$. Observe

$$\begin{aligned} x &= \frac{y+w}{y-w} \\ x+1 &= \frac{2y}{y-w} \\ w^2 &= y - \sqrt{y} \\ w^4 &= y^2 + y - 2y\sqrt{y}. \end{aligned}$$

Then

$$\begin{aligned} \frac{(x+1)^4}{x(2016x^2 - 2x + 2016)} &= \frac{2^4 y^4}{(y-w)^4} \cdot \frac{y-w}{y+w} \cdot \frac{1}{2016\left(\frac{y+w}{y-w}\right)^2 - 2\left(\frac{y+w}{y-w}\right) + 2016} \\ &= \frac{2^4 y^4}{2016(y+w)^3(y-w) - 2(y+w)^2(y-w)^2 + 2016(y+w)(y-w)^3} \\ &= \frac{2^4 y^4}{2(2015y^4 + 2y^2w^2 - 2017w^4)} \\ &= \frac{2^3 y^3}{2015y^3 + 2yw^2 - w^4} \qquad \text{using } y = 2017 \\ &= \frac{8y^3}{2015y^3 + 2y(y - \sqrt{y}) - (y^2 + y - 2y\sqrt{y})} \\ &= \frac{8y^3}{2015y^3 + y^2 - y} \\ &= \frac{8y^2}{2015y^2 + y - 1} \end{aligned}$$

so that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{8(2017)^2}{2015(2017)^2+2016} = \frac{32546312}{8197604351}.$$

Solution 4 by Arkady Alt, San Jose, CA

Let $x = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}}$. Then, $x+\dfrac{1}{x} = \dfrac{a+\sqrt{a-\sqrt{a}}}{a-\sqrt{a-\sqrt{a}}} + \dfrac{a-\sqrt{a-\sqrt{a}}}{a+\sqrt{a-\sqrt{a}}} =$

$\dfrac{\left(a+\sqrt{a-\sqrt{a}}\right)^2+\left(a-\sqrt{a-\sqrt{a}}\right)^2}{a^2-a+\sqrt{a}} = \dfrac{2\left(a^2+a-\sqrt{a}\right)}{a^2-a+\sqrt{a}} = \dfrac{2\left(-a^2+a-\sqrt{a}+2a^2\right)}{a^2-a+\sqrt{a}} =$

$-2+\dfrac{4a^2}{a^2-a+\sqrt{a}} \iff x+\dfrac{1}{x}+2 = \dfrac{4a^2}{a^2-a+\sqrt{a}}$ and, therefore,

$\dfrac{(x+1)^4}{x((a-1)\,x^2-2x+(a-1))} = \dfrac{(x+1)^4}{x^2((a-1)\left(x+\dfrac{1}{x}+2\right)-2a)} =$

$\dfrac{\left(x+\dfrac{1}{x}+2\right)^2}{(a-1)\left(x+\dfrac{1}{x}+2\right)-2a} = \dfrac{\left(\dfrac{4a^2}{a^2-a+\sqrt{a}}\right)^2}{(a-1)\cdot\dfrac{4a^2}{a^2-a+\sqrt{a}}-2a} =$

$\dfrac{16a^4}{((a-1)\cdot 4a^2-2a\,(a^2-a+\sqrt{a}))\,(a^2-a+\sqrt{a})} = \dfrac{16a^4}{2a\,(a^2-a-\sqrt{a})\,(a^2-a+\sqrt{a})} =$

$\dfrac{8a^3}{(a^2-a)^2-a} = \dfrac{8a^2}{a\,(a-1)^2-1}.$

For $a=2017$ we get $\dfrac{(x+1)^4}{x(2016x^2-2x+2016)} = \dfrac{8\cdot 2017^2}{2017\cdot 2016^2-1}$.

Solution 5 by Kee-Wai Lau, Hong Kong, China

We show that

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} = \frac{32546312}{8197604351} \qquad (1)$$

Firstly we have

$$\begin{aligned} x+\frac{1}{x} &= \frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}} + \frac{2017-\sqrt{2017-\sqrt{2017}}}{2017+\sqrt{2017-\sqrt{2017}}} \\ &= \frac{\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2}{\left(2017-\sqrt{2017-\sqrt{2017}}\right)^2+\left(2017+\sqrt{2017-\sqrt{2017}}\right)^2} \\ &= \frac{2(4070306-\sqrt{2017})}{4066272+\sqrt{2017}} \\ &= \frac{2(4070306-\sqrt{2017})(4066272-\sqrt{2017})}{(4066272+\sqrt{2017})(4066272-\sqrt{2017})} \end{aligned}$$

$$= \frac{2(8205736897 - 4034\sqrt{2017})}{8197604351}.$$

Next, we have

$$\left(x + \frac{1}{x} + 2\right)^2 = \frac{131291822608(8197604353 - 4032\sqrt{2017})}{67200717095534131201}$$

and

$$2016\left(x + \frac{1}{x}\right) - 2 = \frac{4034(8197604353 - 4032\sqrt{2017})}{8197604351}.$$

Since $\dfrac{(x+1)^4}{x(2016x^2 - 2x + 2016)} = \dfrac{\left(x + \frac{1}{x} + 2\right)^2}{2016\left(x + \frac{1}{x}\right) - 2}$, so (1) follows.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Telman Rashidov, Azerbaijan Medical University, Baku Azerbaijan; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5428: *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

If $x > 0$, then $\dfrac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} + \dfrac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} \geq 1 - \dfrac{1}{\sqrt[4]{2}}$, where [.] and {.} respectively denote the integer part and the fractional part of x.

Solution 1 by Soumava Chakraborty, Kolkata, India

Case 1: $0 < x < 1 \quad [x] = 0$. Therefore,

$$LHS = \frac{\{x\}}{\sqrt[4]{17\{x\}^4}} = \frac{1}{\sqrt[4]{17}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 2: $[x] \geq 1$ and $\{x\} = 0$. Therefore,

$$LHS = \frac{[x]}{\sqrt[4]{2[x]^4}} = \frac{1}{\sqrt[4]{2}} > 1 - \frac{1}{\sqrt[4]{2}}.$$

Case 3: $[x] \geq 1$ and $0 < \{x\} < 1$. Therefore,

$$\{x\} < 1 \leq [x] \quad \Rightarrow \quad \{x\} < [x]\Big(2\{x\} + [x]\Big)^4 + [x]^4 < 82[x]^4$$

$$\Rightarrow \quad \frac{[x]}{\sqrt[4]{[x]^4 + ([x] + 2\{x\})^4}} > \frac{1}{\sqrt[4]{82}}, \text{and } \frac{\{x\}}{\sqrt[4]{\{x\}^4 + ([x] + 2\{x\})^4}} > 0, \text{ and therefore}$$

$$LHS \quad > \quad \frac{1}{\sqrt[4]{82}} > 1 - \frac{1}{\sqrt[4]{2}}.$$